Lecture 6 - Holomorphic Dynamics

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational

$\Omega(f) := \{ z \in \hat{\mathbb{C}} : \exists\, \mathcal{U} \ni z : (f^n|_{\mathcal{U}})_{n \geq 0}$ is normal $\}$

FATOU SET

$J(f) := \hat{\mathbb{C}} \setminus \Omega(f)$

JULIA SET

$J(f) \neq \emptyset$

ATTRACTING PERIODIC PTS belong to $\Omega(f)$
REPELLING PER. PTS belong to $J(f)$
PARABOLIC PER. PTS belong to $J(f)$

$GO(z, f) = \{ w : \exists\, m, n \geq 0 : f^m(w) = f^n(z) \}$

Def.: A periodic point z is <u>superattracting</u> if $(f^p)'(z) = 0$
(where p is s.t. $f^p(z) = z$)

Lemma If $\deg f \geq 2$, then there at most 2 points which have finite grand orbit. Moreover, such points must belong to superattracting cycles.

Proof $f(GO(z,f)) = GO(z,f)$

hence, if $GO(z,f)$ is finite,

$$f|_{GO(z,f)} \text{ is a bijection.}$$

so $GO(z,f)$ must be a cycle:

$$\begin{tikzcd} \circ \arrow[r, "d"] & \circ \arrow[r, "d"] & \circ \arrow[r, "d"] & \circ \arrow[lll, bend left, "d"] \end{tikzcd}$$

so $\deg f|_{z'} = d = \deg f$ at every $z' \in GO$.

hence all such points are critical, so $GO(z,f)$ is a superattracting cycle.

If $\# GO(z,f) \geq 3$, then

$$\hat{\mathbb{C}} \setminus GO(z,f) \text{ is a hyperbolic R.S.}$$

Then: $f^n : \hat{\mathbb{C}} \setminus GO \to \hat{\mathbb{C}} \setminus GO$

is normal by Montel. Hence both $\hat{\mathbb{C}} \setminus GO$ and GO would be subsets of $\Omega(f)$

hence $J(f) = \emptyset$, contradiction.

Rmk.: $f(z) = 2ze^z$ does have a grand orbit finite point which is not superattracting (in fact, repelling)

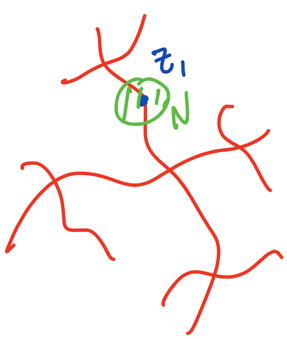
Thm (transitivity)

Let $z_1 \in J(f)$ and let $N \ni z_1$ a neighborhood. Then the union

$$\mathcal{U} = \bigcup_{n=0}^{\infty} f^n(N)$$

contains all $\hat{\mathbb{C}}$ except at most 2 points. If N is small enough, then $\mathcal{U} = \hat{\mathbb{C}} \setminus \mathcal{E}(f)$ where $\mathcal{E}(f)$ is the set of points with finite grand orbit.

Pf



If $|\hat{\mathbb{C}} \setminus \mathcal{U}| \geq 2$,

$$f(\mathcal{U}) \subseteq \mathcal{U}$$

hence $\mathcal{U} \subseteq \Omega(f)$ by Montel

but this contradicts $z_1 \in J \cap N$.

Every preimage of some $z \in \hat{\mathbb{C}} \setminus \mathcal{U}$ is again contained in $\hat{\mathbb{C}} \setminus \mathcal{U}$, hence z is grand orbit finite.

$$\hat{\mathbb{C}} \setminus \mathcal{U} \subset \mathcal{E}(f)$$

If we pick N small enough so that $N \subset \hat{\mathbb{C}} \setminus \mathcal{E}(f)$, then

$$\hat{\mathbb{C}} \setminus \mathcal{U} \supset \mathcal{E}(f). \qquad \checkmark.$$

Cor.: The Julia set does not contain any open subset of $\hat{\mathbb{C}}$ unless $J(f) = \hat{\mathbb{C}}$.

Pf.: Suppose $\mathcal{U} \subset J(f)$.

Then $$\bigcup_{n=0}^{\infty} f^n(\mathcal{U}) \subset J(f)$$

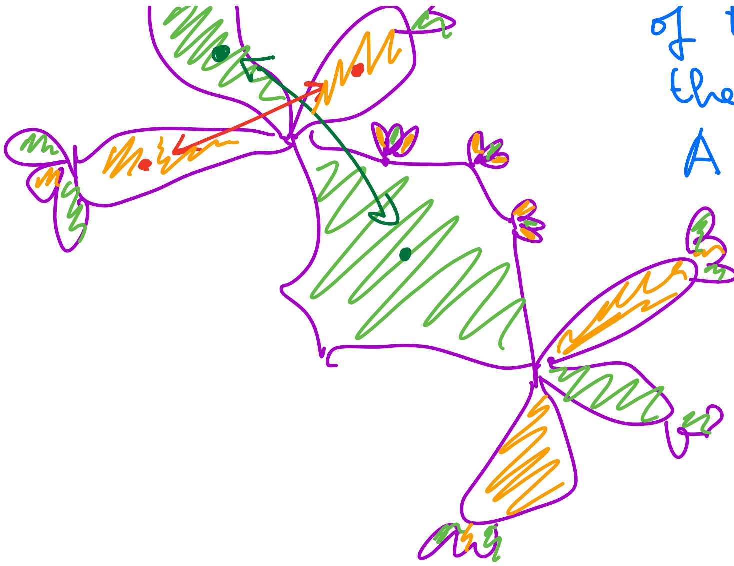
$$\| \text{ transitive}$$

$$\hat{\mathbb{C}}.$$

Cor.: If $\mathcal{A} \subset \hat{\mathbb{C}}$ is the basin of attraction of a periodic cycle, then

$$\partial \mathcal{A} := \overline{\mathcal{A}} \setminus \mathcal{A} = J(f)$$

Rk.: the closure



of the UNION of the component of A may be BIGGER than than the union of the closures.

Pf.: If $N \ni z_1$, $z_1 \in J(f)$

then $f^n(N) \cap A \neq \emptyset$ for some n

hence $N \cap A \neq \emptyset \Rightarrow J \subset \bar{A}$

but $J \cap A = \emptyset \Rightarrow J \subset \partial A$.

$\partial A \subset J$ If $z \in \partial A$, for any N

$f^n|_N$ is not normal $\Rightarrow z \in J(f)$.

Cor.: If $z_0 \in J(f)$, then

$\{z \in \hat{\mathbb{C}} : f^n(z) = z_0 \text{ for some } n \geq 0\}$

is everywhere dense in $J(f)$

Pf: $z_0 \in J(f)$, then $z_0 \notin \mathcal{E}(f)$

$z_1 \in J(f)$